A gentle introduction to Logarithms

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This is a short “explainer” on Logarithms for my daughter, Georgie[1], to help prepare her for A level maths. I’ve written it to deepen understanding and provide a powerful tool.

We will learn that a logarithm (written as “log”, “lg”, “ln”, $\log_3$" etc) is the inverse of raising to a power (e.g. 10^6, 2^8, $e^{-2.312}$, 3^3 etc). But there are some subtle aspects, so we will take our time to build up to this point, and learn some important things along the way.

To start, we will develop intuition by explaining and motivating some of the properties of powers. We can then understand the rules of logarithms (“inverse powers”) from the rules of powers (“inverse logarithms”).

This will take us quite a long way, but it has some limitations. For example, it won’t easily help us to understand e or natural logarithms (written “ln”). So in Chapter 4 we will turn everything on its head and start again from a function with a particular property, use that to *define* the power, b^x, and hence derive logarithms, the number e, and all the key results. We will see that this approach is completely consistent with the power rules and logarithm rules we developed before, but cleaner and with fewer restrictions. It will also prepare us for more advanced ideas such as what we might mean by expressions like $e^{i\pi}$ where $i = \sqrt{-1}$, although we won’t actually learn about such complex numbers here.

The most important outcome is a sense of familiarity, feeling comfortable with the structure of powers and logarithms, having confidence using the syntax, and being aware of the “gotchas”.

I will give several “pit stop” summaries along the way, and finish by providing a complete summary as a reference (or “cheat sheet”). Safe travels, and I hope you enjoy the journey :-)

[1] Who was 16 years old at the time this was written.

Contents

1. Integer Powers

In this chapter, we will start by working out a rule for multiplying positive integer powers. Then we will do the same for dividing, use that to motivate what raising to a *negative* power might mean, and then show that we can use the *same* rule for both positive and negative powers.

We finish by discussing what this tells us about raising a number to the power 0.

1.1. Multiplying positive integer powers

We begin by multiplying different *positive* integer powers with the same base. For example, if we use a base of 2, we can see why multiplying "2 cubed" by "2 squared" equals 2 to the power of 5 ($5 = 2 + 3$):

$$\begin{aligned} 2^3 \times 2^2 &= (2 \times 2 \times 2) \times (2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^{3+2} \end{aligned}$$

It is easy to see that this is true in general for other powers, so for any two positive integers x and y we have:

$$2^x \times 2^y = 2^{x+y}$$

There is nothing special about using base 2. We could have used base 10 or indeed any positive integer base b, *so long as we are consistent about using the same base throughout.* For example:

$$5^x \times 5^y = 5^{x+y}$$

But there is *no* such obvious rule if the bases are different:

$$3^x \times 8^y = ???$$

So in general, for positive integer base b, and any two positive integers x and y, we have a rule which in a sense *turns multiplication into addition*:

$$b^x \times b^y = b^{x+y} \tag{1}$$

1.2. Dividing positive integer powers

Let's repeat the first example above, but using division rather than multiplication:

$$
\begin{aligned}
\frac{2^3}{2^2} &= \frac{2 \times 2 \times 2}{2 \times 2} \\
&= \frac{\cancel{2} \times \cancel{2} \times 2}{\cancel{2} \times \cancel{2}} \\
&= 2^{3-2}
\end{aligned}
$$

I think it is clear that this works for any positive integer base b and any two positive integers x and y. It produces a rule which in a sense *changes division into subtraction*:

$$
\frac{b^x}{b^y} = b^{x-y} \tag{2}
$$

Hang on! What happens if $x < y$, so $(x - y)$ is negative? Do we have to say our rule doesn't work if $x < y$? What could it mean to raise a number to a *negative* power? What is 5 "times by itself *negative* 3 times", 5^{-3}?

1.3. Negative powers

One of the many delightful things about maths is that we get to decide what things mean. Typically, we want rules which make things consistent and general. Why have special cases if we can avoid them?[2]

Consider another example: $x = 2$ and $y = 4$. Note that $x < y$.

$$
\begin{aligned}
\frac{2^2}{2^4} &= \frac{2 \times 2}{2 \times 2 \times 2 \times 2} \\
&= \frac{\cancel{2} \times \cancel{2}}{\cancel{2} \times \cancel{2} \times 2 \times 2} \\
&= \frac{1}{2^2}
\end{aligned}
$$

But according to our rule in Equation 2, this also equals:

$$
\frac{2^2}{2^4} = 2^{2-4} = 2^{-2}
$$

So, in the spirit of deciding the rules to make things consistent, we shall decide that raising to a negative integer means taking reciprocal and then raising to the positive, i.e.

$$
b^{-x} = \frac{1}{b^x} \tag{3}
$$

[2] Is it easier to learn a language with irregular verbs like "choose/chosen" and "drink/drunk", or do you prefer regular verbs like "smile/smiled", "dance/danced", "walk/walked", "sheep/sheeped", … ?

Just two ways of writing the same thing. And now our division rule works for *any* positive integers x and y. Nice.

In fact it gets better. Since we have decided what a negative power means, can we allow x or y (or both) to be negative? Consider another example:

$$\begin{aligned}
&3^{-3} \times 3^2 \\
&= \frac{1}{3^3} \times 3^2 \\
&= \frac{1}{3} \\
&= 3^{-1}
\end{aligned}$$

But look! If we allow negative powers, we could have just used our first rule, Equation 1, to go straight there in one go:

$$3^{-3} \times 3^2 = 3^{-3+2} = 3^{-1}, \qquad \text{because } b^x \times b^y = b^{x+y}$$

1.4. "Pit stop" summary

Let's summarise where we have got to so far.

We have defined a more general *rule* for how to multiply powers and a more general *interpretation* of powers which includes negative integer powers. The interpretation and the rule together go "hand in hand": our rule motivated an interpretation of negative powers, which in turn allowed the rule to be more general purpose, …

Negative integer powers mean reciprocal of the positive power. The words are harder than the equation:

$$b^{-x} = \frac{1}{b^x}$$

Hence we only need one rule to show us how to multiply or divide numbers with the same base, for both positive *and negative powers*:

$$b^x \times b^y = b^{x+y}, \qquad \text{for any integers } x, y \tag{4}$$

If we have a division, then start by writing it using a negative index. For example:

$$\frac{b^x}{b^y} = b^x \times \frac{1}{b^y} = b^x \times b^{-y} = b^{x+(-y)}$$

1.5. Raising to the power 0

We finish this chapter by using our knowledge to work out what it means to raise a number to the power of 0.

First, I don't think anyone would disagree with:

$$\frac{2^2}{2^2} = 1$$
$$\frac{3^9}{3^9} = 1$$
$$\frac{10^{100}}{10^{100}} = 1$$
$$\frac{b^x}{b^x} = 1$$

But if we use Equation 4 then all four of these examples equal some base to the power 0:

$$\begin{aligned}\frac{2^2}{2^2} &= 2^{2-2} &&= 2^0\\ \frac{3^9}{3^9} &= 3^{9-9} &&= 3^0\\ \frac{10^{47}}{10^{47}} &= 10^{47-47} &&= 10^0\\ \frac{b^x}{b^x} &= b^{x-x} &&= b^0\end{aligned}$$

Hence, it is reasonable to say that raising any base b to the power 0 is equal to 1:

$$b^0 = 1 \qquad \text{for any positive integer base, } b \tag{5}$$

(As an alternative derivation, we could start with Equation 4 and let $x = 0$:

$$b^0 \times b^y = b^{0+y} = b^y$$

So b^0 is the number which when multiplied by any number b^y, gives us b^y. Hence $b^0 = 1$.)

What if $b = 0$? What does 0^0 mean, and what does it equal? We could argue that since $0^2 = 0^3 = 0^{100}$ are all equal the 0, then $0^0 = 0$. However this conflicts with the rule that $b^0 = 1$. So we have to decide what we want this to mean in such a way that it fits in with as much of other mathematics as possible.

For example, we would like $0^0 = 1$ so that the following are true statements when $x = 0$:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \qquad \text{at } x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}, \qquad \text{at } x = 0$$

(don't worry if you don't understand what these mean - they are just examples)

Mathematicians usually define $0^0 = 1$. But beware that sometimes it is better to say that it is ambiguous and should be considered *undefined* (much like most mathematicians find that dividing by 0 is best left undefined).

2. Fractional powers

So far, we have a nice rule which turns multiplication of integer[3] powers of a base into adding the powers (see Equation 4).

We will now look at some examples to motivate attaching a meaning to *fractional* powers, such as $2^{\frac{1}{2}}$ and $5^{\frac{2}{3}}$. Hence we will find it natural to expand the scope of the rule above to work with non-integers.

2.1. Power of a power

First, consider:

$$\begin{aligned}(2^3)^2 &= 2^3 \times 2^3\\ &= 2^{3+3}\\ &= 2^6\end{aligned}$$

Another example:

$$\begin{aligned}(5^2)^4 &= 5^2 \times 5^2 \times 5^2 \times 5^2\\ &= 5^{2+2+2+2}\\ &= 5^{2\times 4}\\ &= 5^8\end{aligned}$$

So in general, I think it is clear that:

$$(b^x)^y = b^{x\times y} \qquad \text{for any positive integers } x, y \tag{6}$$

2.2. Reciprocal power

Now let's ask ourselves what it might mean to raise to a reciprocal power. As an example, let $x = 2$ and $y = \frac{1}{2}$. Then using the above rule and ignoring the fact that we said it only worked for integers:

$$(b^2)^{\frac{1}{2}} = b^{2\times\frac{1}{2}} = b^1 = b$$

So whatever raising to the half means[4], it takes the number b^2 back to b: it is the inverse of the function which squares things. Sounds like taking the square root:

$$b = \sqrt{b^2} = (b^2)^{\frac{1}{2}} = b^{2\times\frac{1}{2}} = b^1 = b$$

That all looks reasonable. What if $x = \frac{1}{2}$ and $y = 2$?

[3] Remember that by definition, an integer is a positive or negative whole number (or 0).
[4] b times by itself, half a time…?

$$b = \left(\sqrt{b}\right)^2 = \left(b^{\frac{1}{2}}\right)^2 = b^{\frac{1}{2}\times 2} = b^1 = b$$

Of course! Since $2 \times \frac{1}{2} = \frac{1}{2} \times 2$, the order of the x and y do not matter:

$$(b^x)^y = b^{x\times y} = b^{y\times x} = (b^y)^x$$

But note, we are now allowing the base to be any positive *number,* and not just a positive *integer.* For example:

$$\left(2^{\frac{1}{2}}\right)^2 = \left(\sqrt{2}\right)^2 = b^2, \qquad \text{where the base } b = \sqrt{2} \approxeq 1.4142135624$$

Returning to our $b^{\frac{1}{2}}$, we have concluded that:

$$b^{\frac{1}{2}} = \sqrt{b}$$

What about $b^{\frac{1}{3}}$? Well, all the same reasoning applies:

$$\left(b^{\frac{1}{3}}\right)^3 = b^{\frac{1}{3}\times 3} = b^1 = b$$

So whatever $b^{\frac{1}{3}}$ means, if we cube it then we get back to b: it is the inverse of the function which cubes things, i.e. taking the cube root:

$$b^{\frac{1}{3}} = \sqrt[3]{b}$$

In general, we have decided to interpret raising to the $\frac{1}{x}$ to mean taking the x'th root.

$$b^{\frac{1}{x}} = \sqrt[x]{b}$$

2.3. What do we mean by root, exactly?

What do we mean by root? Do we really know what that means? Well, the square root, cube root, etc are the *inverse* functions of the square, cube, etc. So a better way of defining $b^{\frac{1}{x}}$ is as an inverse. The function we are inverting is a little hidden by the notation, and we need to keep clear in our heads that we are raising some argument b to a fixed x (such as 2 or 3). So b is the function argument we are interested in, not the x (which for now, we consider to be fixed[5]). Let's make all this explicit.

Define the function $f_x(.)$

$$f_x(b) = b^x$$

For example:

[5]Watch this space! Later we will look at the inverse with respect to x, and hence discover logarithms…

$$f_2(b) = b^2$$
$$f_{10}(b) = b^{10}$$

Then we are claiming that the function $g_x(.)$ is the inverse

$$g_x(b) = b^{\frac{1}{x}}$$

Let's check. The definition of inverse is that the inverse function applied to the function returns to the original argument. And conversely, the function applied to the inverse function returns to the original argument.

$$f_x(g_x(b)) = \left(b^{\frac{1}{x}}\right)^x = b^{\frac{x}{x}} = b^1 = b \qquad \checkmark$$
$$g_x(f_x(b)) = (b^x)^{\frac{1}{x}} = b^{\frac{x}{x}} = b^1 = b \qquad \checkmark$$

All is good!

Can you think of a way to write the inverse of f using only f? How about:

$$g_x(b) = f_{\frac{1}{x}}(b)$$

If might help to realise, using our examples from earlier, that

$$f_2(b) = b^2 \qquad \longleftarrow \text{is the inverse of} \longrightarrow \qquad \sqrt{b} = b^{\frac{1}{2}} = f_{\frac{1}{2}}(b)$$
$$f_{10}(b) = b^{10} \qquad \longleftarrow \text{is the inverse of} \longrightarrow \qquad \sqrt[10]{b} = b^{\frac{1}{10}} = f_{\frac{1}{10}}(b)$$

You should check that:

$$f_x\left(f_{\frac{1}{x}}(b)\right) = b, \qquad \text{and}$$
$$f_{\frac{1}{x}}(f_x(b)) = b$$

2.4. Fractional powers

We are now ready to talk about raising a base to a positive fraction where the numerator is not 1. Let's write a fraction as $\frac{x}{y}$, where x and y are both positive integers. We proceed by unpacking the fraction into a product, $\frac{x}{y} = x \times \frac{1}{y}$, hence

$$b^{\frac{x}{y}} = b^{x \times \frac{1}{y}} = (b^x)^{\frac{1}{y}} = \sqrt[y]{b^x} \qquad \text{or,}$$
$$b^{\frac{x}{y}} = b^{\frac{1}{y} \times x} = b^{\left(\frac{1}{y}\right)^x} = \left(\sqrt[y]{b}\right)^x$$

Let's practice with another example. What happens if we raise $b^{\frac{x}{y}}$ to the power y?

$$\left(b^{\frac{x}{y}}\right)^y = b^{\frac{x}{y} \times y} = b^x$$

Does that seem reasonable?

2.5. Combining fractional with negative power

Can we combine this with negative powers from before? What does $b^{-\frac{x}{y}}$ mean? Well, just take it step by step. First the negative power means[6] reciprocal, and the denominator of the fraction just means root. E.g.

$$b^{-\frac{x}{y}} = \frac{1}{b^{\frac{x}{y}}} = \frac{1}{\sqrt[y]{b^x}} \qquad \left(\text{or } \frac{1}{\left(\sqrt[y]{b}\right)^x}\right)$$

Can you see the error in thinking that perhaps

$$b^{-\frac{x}{y}} \stackrel{?}{=} b^{\frac{y}{x}}$$

What is being confused here?

It might all look complicated, but really that's because the syntax is a little messy. The ideas are straightforward, and motivated by the desire to have general purpose[7] rules for combining powers.

2.6. "Pit stop" summary

It is time for another pit stop and to summarise this leg of our journey. Where are we? How did we get here?

We started off by reasoning that

$$(b^x)^y = b^{x \times y} \qquad \text{for any integers } x, y$$

Then we realised that if we interpreted a reciprocal index $\frac{1}{x}$ to mean the x'th root,

$$b^{\frac{1}{x}} = \sqrt[x]{b}$$

we could remove our restriction to integers and declare the rule valid for all rational numbers (numbers which can be written as a ratio). Hence:

$$(b^x)^y = b^{x \times y} \qquad \text{for any rationals } x, y$$

Does the *structure* of this approach feel familiar? It should, because it is analogous to how in the first section we started with

$$b^x \times b^y = b^{x+y}, \qquad \text{for positive integers } x, y$$

[6]Remember: we decided on these rules…

[7]No irregular verbs, please!

Then we realised that if we allowed a negative power to mean reciprocal,

$$b^{-x} = \frac{1}{b^x}$$

we could remove our restriction to only *positive* integers, and hence declare the rule valid for *all* integers (positive and negative).

$$b^x \times b^y = b^{x+y}, \qquad \text{for all integers } x, y$$

In both cases, we started with a very reasonable rule for combining powers which was restricted to a certain type of number; and that motivated us to interpret a power of a more general type of number.

We can show these structural similarities in a table:

Original rule	**...which motivated the *interpretation* of...**	**...and hence gave rise to the more *general* rule**
$b^x \times b^y = b^{x+y}$ for *positive* integers x, y	*negative* indexes: $b^{-x} = \frac{1}{b^x}$	$b^x \times b^y = b^{x+y}$ for *all* integers x, y
$(b^x)^y = b^{x \times y}$ for any *integers* x, y	*reciprocal* indexes: $b^{\frac{1}{x}} = \sqrt[x]{b}$	$(b^x)^y = b^{x \times y}$ for any *rationals* x, y

Now that we know what fractional indexes mean, can we return to $b^x \times b^y = b^{x+y}$ and ask if it also works for fractional indexes as well. The answer is yes, and we can prove it.

Since x and y are both fractions, they will have a common denominator, d. So we can write

$$x = \frac{\hat{x}}{d} \qquad \text{and} \qquad y = \frac{\hat{y}}{d}$$

Hence:

$$
\begin{aligned}
b^x \times b^y &= b^{\frac{\hat{x}}{d}} \times b^{\frac{\hat{y}}{d}} \\
&= \left(b^{\frac{1}{d}}\right)^{\hat{x}} \times \left(b^{\frac{1}{d}}\right)^{\hat{y}} \\
&= \left(b^{\frac{1}{d}}\right)^{\hat{x}+\hat{y}} \\
&= b^{\frac{\hat{x}+\hat{y}}{d}} \\
&= b^{\frac{\hat{x}}{d}+\frac{\hat{y}}{d}} \\
&= b^{x+y} \qquad\qquad \square
\end{aligned}
$$

Let's bring everything together.

Both our rules work with positive and negative fractions (i.e. rationals):

$$
\left.\begin{aligned}
b^x \times b^y &= b^{x+y} \\
(b^x)^y &= b^{x \times y}
\end{aligned}\right\} \text{ for any rationals } x, y \tag{7}
$$

On the way, this led us to attach meaning to negative and fractional powers:

$$
\begin{aligned}
b^{-x} &= \frac{1}{b^x} \\
b^{\frac{1}{x}} &= \sqrt[x]{b}
\end{aligned} \tag{8}
$$

Rather than continuing down this path of developing these rules for even more general class of number, we are going to stop there. Anyway, can't all numbers be written as a fraction? You may be surprised to learn that the answer is no: there are so-called "irrational" numbers which cannot be written as a ratio; even a seemingly familiar number like $\sqrt{2}$. Curious? See Appendix A.

But we aren't going to be distracted by these "technicalities". We have built some good intuition for the rules to combine powers. It's nearly time to talk about logarithms.

2.7. Did we need those brackets?

Before we move on, you might have wondered about the brackets in Equation 6, which I'll repeat again here for convenience:

$$
(b^x)^y = b^{x \times y} \qquad \text{for any integers } x, y
$$

Are the brackets around the b^x necessary? To help decide, think about what the differences might be between the following

$$
b^{(x^y)} \qquad ? \qquad (b^x)^y
$$

Does it even matter? Let's explore by picking a few numbers and evaluating:

$$\begin{aligned} 3^{(2^4)} &= 3^{16} = 43046721 \\ \left(3^2\right)^4 &= 9^4 \qquad = 6561 \end{aligned}$$

So yes, the brackets do matter because $43046721 \neq 6561$.

What if we don't use any brackets? What do we want b^{x^y} to mean? What is 3^{2^4}? Mathematicians like to write things in a concise way, avoiding extra marks on the page. Hence they define the version without brackets to mean the most commonly used expression, which happens to be $b^{(x^y)}$ e.g. $3^{(2^4)}$. If we want the other then we must use brackets to say so. In short,

$$b^{x^y} = b^{(x^y)} \neq (b^x)^y$$

Hence the brackets in Equation 6 were necessary. (If in doubt, use brackets to remove ambiguity.)

3. Logarithms as an inverse power

We are now ready to talk about logarithms. To help us see the structure and symmetry, we will start by writing "raise a number to a power" as a *function.* Then we will define logarithm as the inverse of this function, before showing how our power rules map into analogous rules for logarithms. We will end this chapter by asking if there is a best or most "natural" base.

3.1. Writing power as a function

Let $\mathrm{pow}_b(.)$ be the function which raises a given b to the power of its argument:

$$\mathrm{pow}_b(x) = b^x$$

This is just writing the same thing in a different way - a change of syntax. How do we write our rules[8] in this syntax?

$$\begin{aligned} b^x \times b^y = b^{x+y} &\qquad \longleftrightarrow \qquad \mathrm{pow}_b(x) \times \mathrm{pow}_b(y) = \mathrm{pow}_b(x+y) \\ (b^x)^y = b^{x \times y} &\qquad \longleftrightarrow \qquad [\mathrm{pow}_b(x)]^y = \mathrm{pow}_b(x \times y) \end{aligned} \tag{9}$$

3.2. Defining logarithm as the inverse power

Now for the key *question*:

What is the inverse[9] function of $\mathrm{pow}_b(.)$? What is ${\mathrm{pow}_b}^{-1}(.)$?

And the *answer* is:

The **logarithm** in base b, written $\log_b(.)$

Let's be really explicit to show the symmetry:

$$\log_b(x) = {\mathrm{pow}_b}^{-1}(x)$$
$$\mathrm{pow}_b(x) = {\log_b}^{-1}(x)$$

Of course, this means that

$$\mathrm{pow}_b(\log_b(x)) = x$$
$$\log_b(\mathrm{pow}_b(x)) = x$$

It's helpful to see the shape of these two functions as graphs:

[8] Remember, these rules work for all rational numbers (both positive and negative fractions).

[9] Be careful not to confuse the notation for inverse function with the notation for raising to the power of minus one (i.e. taking the reciprocal): $f^{-1}(x) \neq f(x)^{-1}$!

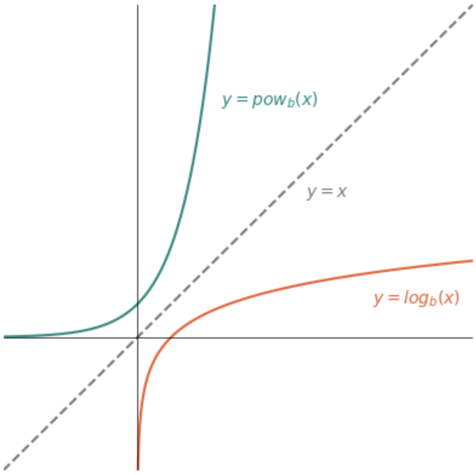

Figure 1: Graphs of $y = \log_b(x)$ and $y = \text{pow}_b(x)$.

Let's make a few general observations:

- $y = \text{pow}_b(x)$ crosses the y axis at $y = 1$.
- $y = \log_b(x)$ crosses the x axis at $x = 1$.
- The $\text{pow}_b(.)$ function is entirely in the top half.
- The $\log_b(.)$ function is entirely in the right half.

Each of the following is true because the two functions are inverses of one another:

- We have symmetry about the line $y = x$.
- The domain[10] of $\text{pow}_b(.)$ equals the range[11] of $\log_b(.)$.
- The range of $\text{pow}_b(.)$ equals the domain of $\log_b(.)$.

3.3. Rules for logarithms

Let us now see if we can take our rules for combining powers (e.g. Equation 9) and find the corresponding rules for logarithms.

Start by defining a change of variable X and Y:

$$\begin{aligned} X &= \text{pow}_b(x), \qquad \text{and hence} \qquad x = \log_b(X) \\ Y &= \text{pow}_b(y), \qquad \text{and hence} \qquad y = \log_b(Y) \end{aligned}$$

Then by taking logarithms and switching to X and Y, we find we can convert our first power rule into a logarithm rule:

[10] The domain of a function is the set of all possible *inputs*.

[11] The range of a function is the set of all possible *outputs*.

$$
\begin{aligned}
& \operatorname{pow}_b(x) \times \operatorname{pow}_b(y) = \operatorname{pow}_b(x+y) \\
\Rightarrow \quad & \log_b(\operatorname{pow}_b(x) \times \operatorname{pow}_b(y)) = \log_b(\operatorname{pow}_b(x+y)) \\
\Rightarrow \quad & \log_b(X \times Y) = x + y \\
\Rightarrow \quad & \log_b(X \times Y) = \log_b(X) + \log_b(Y)
\end{aligned}
$$

Or in words: "**Logarithm changes a product into a sum**". I think that's pretty neat, and turns out to be very powerful.

What about the other rule? The approach is much the same, although we only change one of the variables:

$$
\begin{aligned}
& [\operatorname{pow}_b(x)]^y = \operatorname{pow}_b(x \times y) \\
\Rightarrow \quad & \log_b([\operatorname{pow}_b(x)]^y) = \log_b(\operatorname{pow}_b(x \times y)) \\
\Rightarrow \quad & \log_b(X^y) = x \times y \\
\Rightarrow \quad & \log_b(X^y) = \log_b(X) \times y
\end{aligned}
$$

Or in words: "**Logarithm changes a power into a multiplication**".

It's worth noting that X is in the range of $\operatorname{pow}_b(.)$ i.e. any positive number, but y is in the range of $\log_b(.)$ i.e. any number (positive or negative).

In a sense, logarithms *simplify.* The log function turns are more difficult operation into an easier one: multiplication becomes addition, and raising to a power becomes multiplication. Before we had electronic calculators and computers, books containing tables of logarithms were published to help people perform complex calculations. Thankfully we don't need those any more, but logarithms are still important in maths, engineering, and science, so well worth understanding.

3.4. The power function has *two* inverses

If you look back at Section 2.3 and Section 3.2 then you will notice that we talked about the inverse of the power function in two completely different ways. How can that be?

The power function b^x is a function of *two* things: the base b and the power[12] x. Hence there are two inverses, one for each, with the other being considered as a constant or "given". In fact we can think of the function for b^x in *three* different ways:

$$
\begin{aligned}
f(b, x) &= b^x && \longleftarrow \text{a function of two variables} \\
f_b(x) &= b^x && \longleftarrow \text{a function of one variable } x, \text{ for given } b \\
f_x(b) &= b^x && \longleftarrow \text{a function of one variable } b, \text{ for given } x
\end{aligned}
$$

[12] "Exponent", "index", or "power". All these words are often used for the same thing.

The first of these, $f(b, x)$, doesn't have an inverse. It takes two numbers and returns a single number. It is not possible to uniquely reverse that process and go from a single number to two numbers. For example, what values of b and x make $b^x = 64$? Well $2^6 = 64$ and $4^3 = 64$ both work for a start. Allowing all numbers (not just integers) means there are infinite number of solutions. So definitely not invertible!

The second of these, $f_b(x)$, has an inverse which we defined in Section 3.2 to be $\log_b(x)$.

The third of these, $f_x(b)$, came about when we were thinking about roots in Section 2.3, and has an inverse equal to $f_{\frac{1}{x}}(b)$.

3.5. "Pit stop" summary

Once again, let's take a "pit stop". This one is quick.

We have defined the function $\log_b(.)$ as the inverse of taking a power, being careful which inverse we are talking about. In particular:

$$y = b^x \qquad \Longleftrightarrow \qquad x = \log_b(y)$$

In words, the function $\log_b(.)$, spoken as "logarithm (or log) base b", tells us what power to raise b to in order to obtain a given number, y.

We then used our power laws to derive analogous logarithm laws:

$$\begin{aligned} b^x \times b^y = b^{x+y} \qquad &\Longrightarrow \qquad \log_b(X \times Y) = \log_b(X) + \log_b(Y) \\ (b^x)^y = b^{x \times y} \qquad &\Longrightarrow \qquad \log_b(X^y) = \log_b(X) \times y \end{aligned} \tag{10}$$

Note x and y can be positive or negative, but X and Y must be positive.

Logarithms turn **multiplication into addition**, and **power into multiplication**.

3.6. Is there a "natural" base?

We haven't discussed any particular bases, b, yet. Is there a base that is "better" or more "natural" than others? Is there a base which is "easier" to work with?

One could for example use $b = 10$, so

$$\begin{aligned} 10^3 &= 1000 \\ \log_{10}(1{,}000{,}000) &= 6 \end{aligned}$$

But this is very “human centred”, favouring base 10 simply because humans have 10 fingers. Is there a *mathematical* favouritism which would lead us to a different base? To help us think about this, let’s look at a different area of maths: geometry, and in particular, angles.

Two familiar units of angle are:

- **degrees**, with 360 degrees[13] sweeping round a full circle.
- **radians**, with 2π radians sweeping round a full circle.

Is it easier to work in radians or in degrees?

First, consider two questions:

1. What is the area of a sector of angle θ?
2. What is the arc length of a sector of angle θ?

The area of a circle is πr^2. Half a circle has an area of half that; a quarter sector has area a quarter of that; and in general, a sector fraction of the circle has an area of that fraction of the total area. That fraction will depend on the unit of angle i.e. degrees or radians.

The area of a sector with angle θ is:

$$\frac{\theta}{360} \times \pi r^2 = \frac{1}{2}\theta r^2 \times \frac{2\pi}{360}, \qquad \text{if } \theta \text{ is in degrees}$$

$$\frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}\theta r^2, \qquad \text{if } \theta \text{ is in radians}$$

The arc length of a whole circle is $2\pi r$ (i.e. the circumference). The arc length of half a circle is half of that; a quarter sector has arc length of a quarter of that; and in general, a sector fraction of a circle has an arc length equal to that fraction of the total arc length (i.e. the circumference). So again, the fraction will depend on the unit used to measure angle.

The arc length of a sector with angle θ is:

$$\frac{\theta}{360} \times 2\pi r = \theta r \times \frac{2\pi}{360}, \qquad \text{if } \theta \text{ is in degrees}$$

$$\frac{\theta}{2\pi} \times 2\pi r = \theta r, \qquad \text{if } \theta \text{ is in radians}$$

In a way, this is not very surprising: circle area and circumference involves a special number, π, so we might expect to find π in a more natural unit of angle.

[13]The ancient Babylonians (circa 1700 BCE) used a sexagesimal (base 60) numeral system. This has given us the modern-day 60 seconds in a minute, 60 minutes in an hour, and 360 degrees in a circle.

Let's consider another example. You know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

But actually this is only true if the angle, x, is in radians. If instead we work in degrees then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \frac{2\pi}{360}\times\cos x, \qquad\qquad (x \text{ in degrees})$$

Once again, we see a factor[14] of $\frac{2\pi}{360}$ if we work in degrees. I think all this suggests that it is mathematically easier to work in radians. Hence we could say that radians are a "natural" unit of angle.

So, in a similar sense, is there a natural base for powers/logarithms? Hold that thought for now, and we will return to it later after we have defined a different special number, e.

[14]Note that $\frac{2\pi}{360}$ is a "conversion factor" between different units: $\theta_{\text{radians}} = \frac{2\pi}{360}\times\theta_{\text{degrees}}$.

4. Starting again!

We are now going to start again with very little, and ultimately create logs and powers. This is going to be rather magical and the most exciting part of our journey.

4.1. Small beginnings...

Say someone gives you a function ln(.), but they only tell you two things about it.

First, that it has the following property:

$$\ln(x \times y) = \ln(x) + \ln(y), \qquad x \geq 0, y \geq 0 \tag{11}$$

Second, that it has an *inverse* function, exp(.), i.e.

$$\ln(\exp(x)) = x \qquad \text{and} \qquad \exp(\ln(x)) = x$$

For now, ln(.) is just a function. You might already know of it as “natural log”, but hold that thought back until later.

What can we do with this starting point?

4.2. Immediate consequences

By setting $y = 1$ in Equation 11, we have

$$\begin{aligned} & \ln(x \times 1) = \ln(x) + \ln(1) \\ \Rightarrow \quad & \ln(x) = \ln(x) + \ln(1) \\ \Rightarrow \quad & \ln(1) = 0 \\ \Rightarrow \quad & 1 = \exp(0), \qquad \text{since} \exp(.) \text{ is the inverse of} \ln(.) \end{aligned}$$

By setting $y = \frac{1}{x}$ in Equation 11, we have

$$\begin{aligned} & \ln(1) = \ln(x) + \ln\left(\frac{1}{x}\right) \\ \Rightarrow \quad & \ln\left(\frac{1}{x}\right) = -\ln(x), \qquad \text{since} \ln(1) = 0 \end{aligned}$$

And by taking exp(.) of both sides,

$$
\begin{aligned}
& \exp\left(\ln\left(\frac{1}{x}\right)\right) = \exp(-\ln(x)) \\
\Rightarrow \quad & \frac{1}{x} = \exp(-\ln(x)) \\
\Rightarrow \quad & \frac{1}{\exp(\ln(x))} = \exp(-\ln(x)) \\
\Rightarrow \quad & \frac{1}{\exp(x')} = \exp(-x'), \qquad \text{for any } x' = \ln(x)
\end{aligned}
$$

Returning to Equation 11 and setting $y = 2$, we have

$$\ln(2x) = \ln(2) + x$$

and so by setting $y = 2^n$, we have

$$
\begin{aligned}
\ln(2^n) &= \ln(2 \times 2^{n-1}) \\
&= \ln(2) + \ln(2^{n-1}) \\
&= \ln(2) + \ln(2 \times 2^{n-2}) \\
&= \ln(2) + \ln(2) + \ln(2^{n-2}) \\
&= 2 \times \ln(2) + \ln(2^{n-2}) \\
&= \ldots \\
&= n \times \ln(2) + \ln(2^0) \\
&= n \times \ln(2) + \ln(1) \\
&= n \times \ln(2)
\end{aligned}
$$

Using this same approach for *any* positive integer a, we have

$$\ln(a^n) = n \ln(a)$$

Lastly, and again returning to Equation 11, but this time with a change of variable, $X = \ln(x)$ and $Y = \ln(y)$,

$$
\begin{aligned}
& \ln(x \times y) = \ln(x) + \ln(y) \\
\Rightarrow \quad & \ln(x \times y) = X + Y \\
\Rightarrow \quad & x \times y = \exp(X + Y) \\
\Rightarrow \quad & \exp(X) \times \exp(Y) = \exp(X + Y)
\end{aligned}
$$

In words, “the product of the exp(.) equals the exp(.) of the sum”.

Let's summarise all of this in one place.

By definition:

$$\begin{aligned}
\ln(x \times y) &= \ln(x) + \ln(y) && \text{(defining property of} \ln) \\
\exp(\ln(x)) &= x && (\exp \text{ is defined as inverse of} \ln) \\
\ln(\exp(x)) &= x && (\exp \text{ is defined as inverse of} \ln)
\end{aligned} \tag{12}$$

From which we have proven:

$$\begin{aligned}
\ln(1) &= 0 \\
\exp(0) &= 1 \\
\ln\left(\frac{1}{x}\right) &= -\ln(x) \\
\exp(-x) &= \frac{1}{\exp(x)} \\
\ln(a^n) &= n\ln(a), \qquad \text{for integer } n > 0 \\
\exp(x+y) &= \exp(x) \times \exp(y)
\end{aligned} \tag{13}$$

4.3. Tests

We still haven't said anything about raising a number to a power[15]. Shortly, we will *define* power in a new way, but one which still aligns with our expectations, intuitions, and results we developed earlier. Let's summarise these expectations, and think of them as *tests*: if our new definition passes these tests then we will be justified both in calling it "raising to a power" and also writing it the same way.

So, however b^x is defined, we want it to pass the following tests:

[15] We haven't mentioned logs yet either; remember, ln is just a function with some properties.

$$
\begin{aligned}
2^0 &= 1 \\
2^2 &= 4 \\
5^3 &= 125 \\
2^{-3} &= 0.125 \\
2^{\frac{1}{2}} &= \sqrt{2} \\
10^{\frac{1}{3}} &= \sqrt[3]{10} \\
b^0 &= 1 \\
b^1 &= b \\
b^{-x} &= \frac{1}{b^x} \\
b^{\frac{1}{x}} &= \sqrt[x]{b} && \text{for positive integer } x \\
b^{x+y} &= b^x \times b^y, && \text{for any } x, y \\
(b^x)^y &= b^{x \times y}, && \text{for any } x, y
\end{aligned}
$$

4.4. Candidate definition of b^x

We are ready to present our candidate definition of b^x. I'm going to write it using a different font as $\mathtt{b^x}$ for now, until we have persuaded ourselves that it should be written normally.

$$\mathtt{b^x} = \exp(x \ln b) \tag{14}$$

Don't panic! I know this looks pretty wild, especially as a definition of something we thought we understood already (just raising a number to a power…?!). But anticipate pleasure from the power[16] of understanding new things (and old things in new ways).

Now let $\log_b(.)$ be the inverse of our candidate definition, $\mathtt{b^x}$. Remember that $\ln(.)$ is the inverse of $\exp(.)$, so let's summarise both of them together to help keep things straight in our minds:

$$
\begin{array}{cccc}
\ln(.) & \longleftarrow \text{is the inverse of} \longrightarrow & \exp(.) & \\
\log_b(.) & \longleftarrow \text{is the inverse of} \longrightarrow & \exp(x \ln b) & = \mathtt{b^x}
\end{array}
$$

Therefore, because $\mathtt{b^x}$ and $\log_b(x)$ are inverse functions,

$$x = \mathtt{b^{\log_b(x)}} \qquad \text{and} \qquad x = \log_b(\mathtt{b^x})$$

Before we evaluate our tests, let's derive some useful results which will help us later. These results will all feel a little bit familiar, but remember that we are still only working with a *candidate* definition for b^x (which we are writing as $\mathtt{b^x}$). A sense of familiarity is good because it suggests that we are working with a good candidate.

[16] Sorry.

First, by taking ln(.) of both sides of our new definition, we find:

$$\begin{aligned} \ln(\mathtt{b^x}) &= \ln(\exp(x \ln b)) \\ &= x \ln b \end{aligned} \tag{15}$$

Second, by starting with one of the inverse identities above, $x = \mathtt{b}^{\mathtt{log_b(x)}}$, taking ln(.) of both sides, and then using Equation 15:

$$\begin{aligned} \ln(x) &= \ln\left(\mathtt{b}^{\mathtt{log_b(x)}}\right) \\ &= \log_b(x) \ln b \end{aligned}$$

Rearrange, and we have an equation for $\log_b(.)$ in terms of the function ln(.):

$$\log_b(x) = \frac{\ln(x)}{\ln(b)} \tag{16}$$

Since b was arbitrary, this is also true for another base, a:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Eliminating $\ln(x)$ from both and we find:

$$\frac{\log_a(x)}{\log_b(x)} = \frac{\ln(b)}{\ln(a)} \tag{17}$$

Do we have enough to work out an expression for $\log_a(\mathtt{b^x})$? Yes:

$$\begin{aligned} \log_a(\mathtt{b^x}) &= \frac{\ln(\mathtt{b^x})}{\ln(a)} && \text{(using Equation 16)} \\ &= \frac{\ln(\exp(x \ln b))}{\ln(a)} && \text{(using Equation 14)} \\ &= x \times \frac{\ln(b)}{\ln(a)} && \text{(ln and exp are inverses)} \\ &\left(= x \times \frac{\log_a(x)}{\log_b(x)} \right. && \left. \text{(using Equation 17)} \right) \end{aligned} \tag{18}$$

4.5. Evaluating our tests

We are now ready to see if our tests from Section 4.3 pass on our candidate definition,

$$\mathtt{b^x} = \exp(x \ln b)$$

In what follows, it will be a very good exercise to work out for yourself which definition (Equation 12) or result (Equation 13) is used in each step.

The first few tests are easy to confirm:

$$\begin{aligned}
\mathtt{2}^0 &= \exp(0 \times \ln 2) = \exp(0) &&= 1 && \checkmark \\
\mathtt{2}^2 &= \exp(2 \ln 2) = \exp\big(\ln(2^2)\big) = 2^2 &&= 4 && \checkmark \\
\mathtt{5}^3 &= \exp(3 \ln 5) = \exp\big(\ln(5^3)\big) = 5^3 &&= 125 && \checkmark \\
\mathtt{2}^{-3} &= \exp(-3 \ln 2) = \frac{1}{\exp(3 \ln 2)} = \frac{1}{2^3} &&= 0.125 && \checkmark
\end{aligned}$$

For the test that $\mathtt{2}^{\frac{1}{2}} = \sqrt{2}$, consider its normal square:

$$\begin{aligned}
\left(\mathtt{2}^{\frac{1}{2}}\right)^2 &= \mathtt{2}^{\frac{1}{2}} \times \mathtt{2}^{\frac{1}{2}} \\
&= \exp\left(\frac{1}{2}\ln 2\right) \times \exp\left(\frac{1}{2}\ln 2\right) \\
&= \exp\left(\frac{1}{2}\ln 2 + \frac{1}{2}\ln 2\right) \\
&= \exp(\ln 2) \\
&= 2 \qquad \checkmark
\end{aligned}$$

Similarly, for $\mathtt{10}^{\frac{1}{3}}$, considered its normal cube:

$$\begin{aligned}
\left(\mathtt{10}^{\frac{1}{3}}\right)^3 &= \mathtt{10}^{\frac{1}{3}} \times \mathtt{10}^{\frac{1}{3}} \times \mathtt{10}^{\frac{1}{3}} \\
&= \exp\left(\frac{1}{3}\ln 10\right) \times \exp\left(\frac{1}{3}\ln 10\right) \times \exp\left(\frac{1}{3}\ln 10\right) \\
&= \exp\left(\frac{1}{3}\ln 10 + \frac{1}{3}\ln 10\right) \times \exp\left(\frac{1}{3}\ln 10\right) \\
&= \exp\left(\frac{1}{3}\ln 10 + \frac{1}{3}\ln 10 + \frac{1}{3}\ln 10\right) \\
&= \exp(\ln 10) \\
&= 10 \qquad \checkmark
\end{aligned}$$

Continuing with the tests…

$$\begin{aligned}
b^0 &= \exp(0 \times \ln(b)) = \exp(0) &&= 1 && \checkmark \\
b^1 &= \exp(1 \times \ln(b)) = \exp(\ln(b)) &&= b && \checkmark
\end{aligned}$$

So far, all the tests were about obtaining the correct *value*. For the next test, we are looking to confirm (or not) if $\mathtt{b}^{-\mathtt{x}} = \frac{1}{\mathtt{b}^{\mathtt{x}}}$, i.e. an equation all in terms of our candidate function:

$$\begin{aligned}
\mathtt{b}^{-\mathtt{x}} &= \exp(-x \ln b) \\
&= \frac{1}{\exp(x \ln b)} \\
&= \frac{1}{\mathtt{b}^{\mathtt{x}}} \qquad \checkmark
\end{aligned}$$

For the test that $\mathtt{b}^{\frac{1}{\mathtt{x}}} = \sqrt[x]{b}$, consider multiplying this by itself x times (remember that x is a positive integer):

$$\begin{aligned}
\left(\mathtt{b}^{\frac{1}{\mathtt{x}}}\right)^{x} &= \mathtt{b}^{\frac{1}{\mathtt{x}}} \times \ldots \times \mathtt{b}^{\frac{1}{\mathtt{x}}} && (x \text{ many times}) \\
&= \exp\left(\frac{1}{x} \ln b\right) \times \ldots \times \exp\left(\frac{1}{x} \ln b\right) && (x \text{ many times}) \\
&= \exp\left(\frac{1}{x} \ln b + \ldots + \frac{1}{x} \ln b\right) && (x \text{ many times}) \\
&= \exp\left(x \times \frac{1}{x} \ln(b)\right) \\
&= \exp(\ln(b)) \\
&= b \qquad \checkmark
\end{aligned}$$

And so we have arrived at the last two tests:

$$\begin{aligned}
\mathtt{b}^{\mathtt{x}+\mathtt{y}} &= \exp((x + y) \ln b) \\
&= \exp(x \ln b + y \ln b) \\
&= \exp(x \ln b) \times \exp(y \ln b) \\
&= \mathtt{b}^{\mathtt{x}} \times \mathtt{b}^{\mathtt{y}} \qquad \checkmark
\end{aligned}$$

And

$$\begin{aligned}
(\mathtt{b}^{\mathtt{x}})^{\mathtt{y}} &= \exp(y \ln(\mathtt{b}^{\mathtt{x}})) \\
&= \exp(y \ln(\exp(x \ln b))) \\
&= \exp(y \times x \ln b) \\
&= \exp((x \times y) \ln b) \\
&= \mathtt{b}^{\mathtt{x} \times \mathtt{y}} \qquad \checkmark
\end{aligned}$$

All our tests have passed! We can declare that our candidate definition $\mathtt{b}^{\mathtt{x}}$ is aligned with our "every day" expectations for how b^x behaves and evaluates. In short:

$$b^x = \mathtt{b}^{\mathtt{x}} = \exp(x \ln b)$$

From now on, we can dispense with using the different font.

4.6. "Pit stop" summary

Think back to where we finished in Chapter 2, and in particular Equation 7 and Equation 8; and where we finished in Chapter 3, and in particular Equation 10.

In this chapter, we have *re*-derived all these rules of powers and logs from a single, small starting point. Further, we have expanded the scope of these rules so that they work for all numbers, not just fractions (i.e. rationals). This is because we defined b^x as $\exp(x \ln b)$, which we can evaluate[17] for any positive number b and any number x.

Given a function ln(.) and its inverse exp(.), such that:

$$\begin{aligned} \ln(x \times y) &= \ln(x) + \ln(y) \\ \exp(\ln(x)) &= x \\ \ln(\exp(x)) &= x \end{aligned} \tag{19}$$

We ***defined*** what we mean by b^x and its inverse $\log_b(x)$:

$$\begin{aligned} b^x &= \exp(x \ln b) \\ \log_b(x) &= \frac{\ln x}{\ln b} \\ \log_b(b^x) &= x \\ b^{\log_b(x)} &= x \end{aligned} \tag{20}$$

From which we have ***derived*** all our rules:

$$\begin{aligned} b^x \times b^y &= b^{x+y} \\ (b^x)^y &= b^{x \times y} \\ b^{-x} &= \frac{1}{b^x} \\ b^{\frac{1}{x}} &= \sqrt[x]{b} \\ \log_b(x \times y) &= \log_b(x) + \log_b(y) \\ \log_b\left(\frac{1}{x}\right) &= -\log_b(x) \\ \log_b(x^y) &= y \log_b(x) \\ \log_b(1) &= 0 \end{aligned} \tag{21}$$

And we have shown that the new approach:
- **gives the same answers as the previous approach,**
- **but is no longer restricted to fractions and integers.**

[17] Actually, we can't evaluate it yet since we have only defined it from its properties, not by providing an expression. We will get to that shortly…

The only restrictions we are left with are:

- We work with a positive base, so b^x only makes sense if $b > 0$.
- Logs are only defined for positive numbers, so need $x > 0$ in $\ln(x)$ and $\log_b(x)$.
- The function exp(.) can take positive or negative numbers.

Take a moment to marvel at what we have achieved, and how everything has come together in a such a tidy and consistent way.

We do however have a few loose ends, including:

1. What actually is the function, ln? Perhaps surprisingly, we have only defined it in terms of the *property* we want it to have, $\ln(x \times y) = \ln(x) + \ln(y)$. Is there an *expression* for $\ln(x)$?
2. What is the derivative of $\ln(x)$? What is the derivative of $\exp(x)$?
3. What is the number e? How does it relate to everything we have learned?
4. Finally, what is the answer to the question posed in Section 3.6: is there a “natural” base?

4.7. What *base* is $\ln(\cdot)$?

Both $\ln(\cdot)$ and $\log_b(\cdot)$ are logarithms, where the b in $\log_b(\cdot)$ is called the base. A natural question is: what is the base of $\ln(\cdot)$? What is the value of b which makes $\ln(x) = \log_b(x)$?

We know (Equation 16), that

$$\log_b(x) = \frac{\ln x}{\ln b}$$

So if we choose the base b such that $\ln(b) = 1$, then

$$\ln(x) = \log_b(x), \qquad \text{if} \ln(b) = 1$$

Since $\exp(\cdot)$ is the inverse of $\ln(\cdot)$,

$$\ln(b) = 1 \qquad \Rightarrow \qquad b = \exp(1)$$

So the answer to our question is

$$\boxed{\ln(\cdot) \text{ is logarithm base } b = \exp(1)} \tag{22}$$

Further, with this special value of b, what does that make b^x?

Starting with our definition of power (Equation 20) we find

$$\begin{aligned} b^x &= \exp(x \ln b) \\ &= \exp(x \times 1) \\ &= \exp(x) \end{aligned}$$

Hence

$$\boxed{\exp(x) = b^x \text{ where the base } b = \exp(1)} \tag{23}$$

What is this special value of b? The answer is e…

4.8. Defining the number, e

Continuing from the previous section, e is defined[18] as the number such that

$$\ln(e) = 1, \qquad \text{i.e.} \qquad e = \exp(1)$$

Using our definition for b^x (Equation 20), and setting $b = e$,

$$e^x = \exp(x \ln(e)) = \exp(x \times 1) = \exp(x)$$

And using our definition for $\log_b(x)$ (Equation 20), again setting $b = e$,

$$\log_e(x) = \frac{\ln(x)}{\ln(e)} = \frac{\ln(x)}{1} = \ln(x)$$

Restating for emphasis, the ln and exp functions are logarithm (base e) and raising e to a number, respectively:

$$\begin{aligned} \ln(x) &= \log_e(x) \\ \exp(x) &= e^x \end{aligned}$$

We can often write e^x instead of $\exp(x)$ and save ourselves[19] from writing the letters "x" and "p". Of course, for clarity, we can use $\exp(\cdot)$ if the argument is long or complex. For example, which is clearer?

$$e^{\frac{x^2}{\sqrt{1+x}}} \qquad \text{or} \qquad \exp\left(\frac{x^2}{\sqrt{1+x}}\right)$$

I think you will agree that the second is easier to read, but they both mean the same thing.

We will continue to write $\ln(x)$ or $\ln x$ (without the brackets), but with renewed understanding that it is both a fundamental building-block and also logarithm base e.

[18] There are other definitions for the same number, but we won't get into those.
[19] Did I mention that mathematicians are lazy?

4.9. But what *is* $\ln(.)$?

The next few sections use some calculus - differentiation and integration - but don't worry if some techniques are unfamiliar; it's fine to skim-read and just get a sense!

Time for "a big reveal". If we define $\ln(x)$ as

$$\ln(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t \tag{24}$$

then it turns out that this satisfies our defining property:

$$\ln(x \times y) = \ln(x) + \ln(y)$$

Let's prove it.

First we will need the fact that an integral between two numbers a and c equals the integral from a to b plus the integral from b to c:

$$\int_a^c \cdot = \int_a^b \cdot + \int_b^c \cdot \tag{25}$$

This is intuitively true by thinking in terms of areas under the curve: the area under the curve from a to c equals the area from a to b plus the area from b to c.

Using this idea with $a = 1$, $b = x$, and $c = xy$, we find

$$\begin{aligned}
\ln(x \times y) &= \int_1^{xy} \frac{1}{t}\,\mathrm{d}t && \text{by definition, Equation 24} \\
&= \int_1^{x} \frac{1}{t}\,\mathrm{d}t + \int_x^{xy} \frac{1}{t}\,\mathrm{d}t && \text{using idea Equation 25} \\
&= \ln(x) + \int_x^{xy} \frac{1}{t}\,\mathrm{d}t && \text{by definition, Equation 24}
\end{aligned}$$

Nearly there... If we can show that the remaining integral on the right equals $\ln(y)$ then we will be done. Using the technique called "integration by substitution", we define a new variable s as $\frac{t}{x}$ (remember that x is a constant as far as the integral is concerned). Hence:

$$\begin{aligned}
&& t &= xs \\
\text{and} && \mathrm{d}t &= x\,\mathrm{d}s
\end{aligned}$$

To work out the new limits for the integral, note that when $t = x$ then $s = 1$ and when $s = xy$ then $s = y$. Hence

$$
\begin{aligned}
\int_x^{xy} \frac{1}{t}\,\mathrm{d}t &= \int_1^y \frac{1}{xs} x\,\mathrm{d}s \\
&= \int_1^y \frac{1}{\cancel{x}s} \cancel{x}\,\mathrm{d}s \\
&= \int_1^y \frac{1}{s}\,\mathrm{d}s \\
&= \ln(y) \qquad \square
\end{aligned}
$$

And so we have proven that Equation 24 satisfies our defining property!

4.10. The derivative of $\ln(x)$ and e^x

Let's start with the derivative of $\ln(x)$. Since differentiation is the inverse of integration, and since we now have an expression for $\ln(x)$ as the integral $\int_1^x \frac{1}{t}\,\mathrm{d}t$, then

$$
\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x} \tag{26}
$$

That was easy. What about the derivative of e^x?

Start with

$$
y = e^x
$$

Since $\ln(x)$ is the inverse of $\exp(e)$, and e^x is just another way of writing $\exp(x)$, we have

$$
x = \ln(y)
$$

Differentiate both sides with respect to x and use the chain rule,

$$
1 = \frac{1}{y} \frac{\mathrm{d}y}{\mathrm{d}x}
$$

Therefore

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = y
$$

But $y = e^x$, so the derivative of e^x is just e^x:

$$
\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x \tag{27}
$$

Take some delight in the simple elegance of these two results, Equation 26 and Equation 27!

4.11. The derivative of $\log_b(x)$ and b^x

We can immediately use the results from the previous section to work out the derivatives of $\log_b(\cdot)$ and b^x.

Starting with our definition for b^x and using the chain rule (b is considered constant),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} b^x &= \frac{\mathrm{d}}{\mathrm{d}x} \exp(x \ln b) \\ &= \exp(x \ln b) \times \ln b \\ &= b^x \times \ln b \end{aligned} \tag{28}$$

So the derivative of b^x is just b^x times by a factor of $\ln b$.

Turning to the derivate of $\log_b(x)$, and again starting with our definition,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \log_b(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\ln x}{\ln b} \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \ln x \times \frac{1}{\ln b} \\ &= \frac{1}{x} \times \frac{1}{\ln b} \end{aligned} \tag{29}$$

So the derivative of $\log_b(x)$ is $\frac{1}{x}$ (i.e. the derivative of $\ln x$) divided by a factor of $\ln b$.

4.12. So is there a natural base or not?

Does the factor of $\ln b$ in the previous section remind you of anything? How about the factor

$$\frac{2\pi}{360}$$

we discussed in Section 3.6 when calculating things with angles? We concluded that if we worked with angles in units of radians then we would not have to "carry around" an extra factor, resulting in neater equations. Hence radians are a "natural" unit of angle.

Returning to logarithms and powers, if we choose $b = \exp(1)$ then $\ln b = 1$, and the derivatives in Equation 28 and Equation 29 simplify:

$$\text{If we work in a base } b \text{ such that } \ln b = 1 \text{ then}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} b^x = b^x \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \log_b(x) = \frac{1}{x}$$

We also saw in Section 4.7 that $\ln(\cdot)$ and $\exp(\cdot)$ were both base b such that $\ln b = 1$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$$

This discussion probably feels a bit circular, and in a way it is because everything connects in a tidy and consistent way. I hope it helps you readily agree that the answer to the question is yes, and that the most natural logarithmic base is base e.

4.13. Coda

Before ending with a final summary, I cannot resist teasing you with an astonishing fact. I used units of angles (radians) involving π as an *analogy* to motivate preferring base e, and avoid having to write extra factors all over the place. But actually, there is a deep mathematical connection between natural logarithms $\ln(\cdot)$, the special number e, the ratio of sides of a triangle, $\sin(\cdot)$ and $\cos(\cdot)$, and π. The connection involves another special number, i, defined as the square root of -1, i.e. $i = \sqrt{-1}$, which moves us into the realm of so-called "complex numbers".

Euler's formula says:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

If we let $\theta = \pi$, then of course $\cos(\pi) = -1$ and $\sin(\pi) = 0$, and this becomes Euler's identity:

$$e^{i\pi} + 1 = 0$$

What a beautiful connection between five fundamental mathematical numbers, $e, \pi, 0, 1, i$.

5. Reference

Conventions

- $\ln(\cdot)$ means *natural* logarithm base e, so $\ln(x) = \log_e(x)$.
- $\log(\cdot)$ means logarithm base 10, any base, or even base e (ambiguous!).
- $\log_b(\cdot)$ means logarithm base b (unambiguous).
- $\lg(\cdot)$ means logarithm base 2 (used in computer science).
- Ways to write the same thing: $\ln^{-1}(x) = \exp(x) = e^x$.

Rules of powers

$$\begin{aligned} b^0 &= 1 \\ b^x \times b^y &= b^{x+y} \\ b^{-x} &= \frac{1}{b^x} \\ b^{\frac{1}{x}} &= \sqrt[x]{b} \\ (b^x)^y &= b^{x \times y} \end{aligned}$$

Logarithm is inverse power

$$\begin{aligned} y = b^x \qquad &\iff \qquad x = \log_b(y) \\ \log_b(b^x) &= x, \qquad \text{any } x \\ b^{\log_b(x)} &= x, \qquad x > 0 \end{aligned}$$

Rules of logarithms

$$\begin{aligned} \log_b(x \times y) &= \log_b(x) + \log_b(y) \\ \log_b(x^y) &= y \times \log_b(x) \\ \log_b(1) &= 0 \\ \log_b\left(\frac{1}{x}\right) &= -\log_b(x) \end{aligned}$$

Convert between bases

$$\begin{aligned} \log_b(x) &= \frac{\ln(x)}{\ln(b)} \\ \log_b(x) &= \frac{\log_a(x)}{\log_a(b)} \end{aligned}$$

Natural logarithm and exponential

$$\begin{aligned} \ln(x) &= \log_e(x) \\ \ln(\exp(x)) &= x \qquad \leftarrow \text{ln and exp are inverses} \\ \exp(\ln(x)) &= x \qquad \leftarrow \text{ln and exp are inverses} \\ e^x &= \exp(x) \qquad \leftarrow \text{same thing} \end{aligned}$$

Rules of natural logs/exponentials

$$\begin{aligned} \ln(1) &= 0 \\ \exp(0) &= 1 \\ \ln(x \times y) &= \ln(x) + \ln(y) \\ \exp(x + y) &= \exp(x) \times \exp(y) \\ \ln\left(\frac{1}{x}\right) &= -\ln(x) \\ \exp(-x) &= \frac{1}{\exp(x)} \end{aligned}$$

Define e and general power

$$\begin{aligned} \ln(e) &= 1 \qquad \leftarrow \text{defines } e \\ b^x &= \exp(x \ln b) \end{aligned}$$

Derivatives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \ln(x) &= \frac{1}{x} \\ \frac{\mathrm{d}}{\mathrm{d}x} e^x &= e^x \\ \frac{\mathrm{d}}{\mathrm{d}x} \log_b(x) &= \frac{1}{x} \times \frac{1}{\ln b} \\ \frac{\mathrm{d}}{\mathrm{d}x} b^x &= b^x \times \ln b \end{aligned}$$

A Irrationality of $\sqrt{2}$

Is it possible to write $\sqrt{2}$? as a fraction? As a ratio of two integers?

We will use "proof by contradiction" to show that the answer is no:
1. We will assume that it is true (i.e. assume it is possible to write $\sqrt{2}$ as a fraction).
2. Then derive a contradiction.
3. And hence conclude that it is false (i.e. it is impossible to write $\sqrt{2}$ as a fraction).

Assume is it true. So there is a such a fraction, which we can write as a ratio of two integers a and b that have *no common factors*:

$$\sqrt{2} = \frac{a}{b}$$

Square both sides and rearrange:

$$a^2 = 2b^2 \tag{30}$$

So a^2 is twice b^2 and therefore a^2 is even. But the square of an odd number is odd, and the square of an even number is even. Hence if a^2 is even, then a must be even.

Since a is even, we can write it as twice some integer k:

$$a = 2k \tag{31}$$

Substitute this into Equation 30 gives us

$$\begin{aligned} & & (2k)^2 &= 2b^2 \\ &\Rightarrow & 2k^2 &= b^2 \\ &\text{or} & b^2 &= 2k^2 \end{aligned}$$

Hence b^2 is even, and like before that means b is even, and hence can be written as twice some integer j:

$$b = 2j \tag{32}$$

So Equation 31 and Equation 32 say that a and b have a common factor of 2. But this is a **contradiction** because we assumed at the start that a and b had *no* common factors.

Hence "proof by contradiction" tells us that the **assumption must be false**: $\sqrt{2}$ cannot be written as a fraction. $\sqrt{2}$ is said to be an *irrational* number.